

ON FRACTIONAL POWERS OF GENERATORS OF FRACTIONAL RESOLVENT FAMILIES

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ABSTRACT. We show that if $-A$ generates a bounded α -times resolvent family for some $\alpha \in (0, 2]$, then $-A^\beta$ generates an analytic γ -times resolvent family for $\beta \in (0, \frac{2\pi-\pi\gamma}{2\pi-\pi\alpha})$ and $\gamma \in (0, 2)$. And a generalized subordination principle is derived. In particular, if $-A$ generates a bounded α -times resolvent family for some $\alpha \in (1, 2]$, then $-A^{1/\alpha}$ generates an analytic C_0 -semigroup. Such relations are applied to study the solutions of Cauchy problems of fractional order and first order.

1. INTRODUCTION

Let A be a closed densely defined linear operator on a Banach space X . The resolvent families were introduced by Da Prato [10] to study Volterra integral equations of the form

$$(1.1) \quad u(t) = f(t) + A \int_0^t a(t-s)u(s)ds.$$

A family $\{R(t)\}_{t \geq 0} \subset B(X)$ is called a resolvent family for A with kernel a if

- (a) $R(0) = I$ and $R(t)$ is strongly continuous;
- (b) $AR(t) \subseteq R(t)A$ for every $t \geq 0$;
- (c) for every $x \in D(A)$,

$$R(t)x = x + \int_0^t a(t-s)R(s)Axd s.$$

It is shown that the problem (1.1) is well-posed (in the sense of [28]) if and only if there is a resolvent family for A . Since a C_0 -semigroup is a resolvent family for its generator with kernel $a_1(t) \equiv 1$, and a cosine operator function is a resolvent family for its generator with kernel $a_2(t) = t$, it is natural to consider the resolvent family with kernel $a_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. Also note the following facts: if A generates a C_0 -semigroup, then the Cauchy problem of first order

$$u'(t) = Au(t), \quad t \geq 0; \quad u(0) = x$$

is well-posed; and if A generates a cosine operator function, then the second order Cauchy problem

$$u''(t) = Au(t), \quad t \geq 0; \quad u(0) = x, \quad u'(0) = y,$$

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is also well-posed. This motivates one to consider the relations between the existence of resolvent family for A with kernel $a_\alpha(t)$ and the well-posedness of some kind of fractional Cauchy problem $D_t^\alpha u(t) = Au(t)$ with proper initial values. Such relation was proved by Bajlekova [3] in 2001. The resolvent family for A with kernel a_α was therefore called α -times resolvent family. For more general resolvent families see [24, 25].

On the other hand, it is well known that if $-A$ generates a bounded cosine function operator, then $-A^{1/2}$ generates an analytic C_0 -semigroup of angle $\pi/2$ (cf. [18]). And it was proved by Yosida in 1960 (cf. [17, 32]) that if T is a bounded C_0 -semigroup on a complex Banach space X , with the generator A , then $-A^\alpha$, $0 < \alpha < 1$, generates an analytic semigroup T_α on X , and T_α is subordinated to T through the Lévy stable density function.

For α -times resolvent family, the questions of interest are:

(Q_1) If $-A$ generates a bounded C_0 -semigroup, does $-A^\alpha$ generate an α -times resolvent family?

(Q_2) If $-A$ generates a bounded α -times resolvent family, does $-A^{1/\alpha}$ generate a C_0 -semigroup?

(Q_3) If $-A$ generates a bounded α -times resolvent family, does $-A^{1/2}$ generate an $\alpha/2$ -times resolvent family?

(Q_4) If $-A$ generates a bounded α -times resolvent family, does $-A^\beta$ also generate an α -times resolvent family for some suitable β ?

(Q_5) If $-A$ generates a bounded α -times resolvent family, does $-A^\beta$ generate a γ -times resolvent family for some suitable β and γ ?

Our first aim in this paper is trying to give answers to the above questions in a unified way. We first note the fact: if $-A$ is the generator of a bounded α -times resolvent family, then A is a sectorial operator (see Section 2 for details). Therefore, it is possible to define the fractional power A^b for $b > 0$. By using the theory of functional calculus for sectorial operators (see [4, 14, 21, 26]), we are able to give positive answers to the questions above. These relations are clarified in Section 3.

The second purpose of this paper is to establish connections between solutions of fractional Cauchy problems and Cauchy problems of first order. Obverse that many phenomena in the theory of stochastic processes, finance and hydrology are recently described through fractional evolution equations, see [6, 7, 30, 33] and the references therein. For example, Zaslavsky [33] introduced the fractional kinetic equation

$$(1.2) \quad \begin{aligned} D_t^\alpha u(t, x) + L_x u(t, x) &= 0, \quad t > 0 \\ u(0, x) &= f(x), \end{aligned}$$

for Hamiltonian chaos, where $\alpha \in (0, 1)$, $-L_x$ is the generator of some continuous Markov process, and D_t^α is understood the Caputo fractional derivative in time (see Section 2). Baeumer and Meerschaert [5], and Meerschaert and Scheffler [27] showed that the fractional Cauchy problem (1.2) is related to a certain class of subordinated stochastic processes. More precisely, Theorem 3.1 in [5] shows that the formula

$$(1.3) \quad u(t, x) = \int_0^\infty v((t/s)^\alpha, x) b_\alpha(s) ds,$$

yields a unique strong solution of (1.2), where b_α is the smooth density of the stable subordinator such that the Laplace transform $\widehat{b_\alpha}(\lambda) = \int_0^\infty e^{-\lambda t} b_\alpha(t) dt = e^{-\lambda^\alpha}$ and

v is the solution of

$$(1.4) \quad \begin{aligned} v'_t(t, x) + L_x v(t, x) &= 0, \quad t > 0 \\ v(0, x) &= f(x). \end{aligned}$$

The formula (1.4) can also be explained by the subordination principle for fractional resolvent family, see Theorem 3.1 in [3] or Lemma 2.9. If the fractional power of L_x , L_x^α , is defined, it is also of interest to know the relations between the solution of (1.4) and that of

$$(1.5) \quad \begin{aligned} D_t^\alpha u(t, x) + L_x^\alpha u(t, x) &= 0, \quad t > 0 \\ u(0, x) &= f(x). \end{aligned}$$

In Section 4 we will give this connection. Moreover, Baeumer, Meerschaert and Nane [6] proved that Eq. (1.2) with $\alpha = 1/2$ and the initial value problem

$$(1.6) \quad \begin{aligned} u'_t(t, x) - L_x^2 u(t, x) + \frac{t^{-1/2}}{\Gamma(1/2)} L_x f(x) &= 0, \quad t > 0 \\ u(0, x) &= f(x), \end{aligned}$$

have the same solution; and (1.2) with $\alpha = 1/3$ and

$$(1.7) \quad \begin{aligned} u'_t(t, x) + L_x^3 u(t, x) + \frac{t^{-2/3}}{\Gamma(1/3)} L_x f(x) - \frac{t^{-1/3}}{\Gamma(2/3)} L_x^2 f(x) &= 0, \quad t > 0 \\ u(0, x) &= f(x), \end{aligned}$$

have the same solution, respectively. Another example is given by Allouba and Zheng [1] and DeBlassie [11], they consider the case that $L_x = -\Delta$, the Laplace operator. Keyantuo and Lizama [19] gave the connections between (1.2) with $\alpha = 1/m$ and ordinary non-homogeneous equations. In Section 4, by analysing the solutions of fractional Cauchy problems directly we can recover the result in [19]. Moreover, we will consider more general fractional Cauchy problem with the fractional order not necessarily a rational number.

Our work is organized as follows. We provide in Section 2 some preliminaries of fractional resolvent families and fractional powers of sectorial operators. And then give positive answers to the questions $(Q_1) - (Q_5)$ in Section 3 and more results of fractional generations are obtained as well. Finally, we discuss the relations of solutions of fractional Cauchy problems and Cauchy problems of first order in Section 4.

2. PRELIMINARIES

Throughout the paper, $(X, \|\cdot\|)$ is a complex Banach space, and $B(X)$ is the space of all bounded linear operators on X . A is a closed linear operator on X . We assume throughout this paper that A is densely defined. By $D(A)$, $R(A)$, $\rho(A)$, $\sigma(A)$ and $R(\lambda, A)$ ($\lambda \in \rho(A)$) we denote the domain, range, resolvent set, spectrum set and resolvent of the operator A , respectively.

Recall the Caputo fractional derivative of order $\alpha > 0$

$$D_t^\alpha f(t) := J_t^{m-\alpha} \frac{d^m}{dt^m} f(t),$$

where m is the smallest integer greater than or equal to α , and the Riemann-Liouville fractional integral of order $\beta > 0$

$$J_t^\beta f(t) = g_\beta * f(t) := \int_0^t g_\beta(t-s)f(s)ds,$$

where

$$g_\beta(t) := \begin{cases} \frac{t^{\beta-1}}{\Gamma(\beta)}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

Set moreover $g_0(t) := \delta(t)$, the Dirac delta-function. For details in fractional calculus, we refer the reader to [20, 29].

The Mittag-Leffler function is defined by

$$(2.1) \quad E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \quad \alpha, \beta > 0, z \in \mathbb{C},$$

where the path C is a loop which starts and ends at $-\infty$, and encircles the disc $|t| \leq |z|^{1/\alpha}$ in the positive sense. $E_\alpha(z) := E_{\alpha,1}(z)$. The Mittag-Leffler function $E_\alpha(t)$ satisfies the fractional differential equation

$$D_t^\alpha E_\alpha(\omega t^\alpha) = \omega E_\alpha(\omega t^\alpha).$$

The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$(2.2) \quad \int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad \operatorname{Re} \lambda > \omega^{1/\alpha}, \omega > 0$$

and with their asymptotic expansion as $z \rightarrow \infty$. If $0 < \alpha < 2$, $\beta > 0$, then

$$(2.3) \quad E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), \quad |\arg z| \leq \frac{1}{2}\alpha\pi,$$

$$(2.4) \quad E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z), \quad |\arg(-z)| < (1 - \frac{1}{2}\alpha)\pi,$$

where

$$\varepsilon_{\alpha,\beta}(z) = - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N})$$

as $z \rightarrow \infty$, and the O -term is uniform in $\arg z$ if $|\arg(-z)| \leq (1 - \alpha/2 - \epsilon)\pi$. It is also of interest to know the relations between the Mittag-Leffler function and function of Wright type:

$$E_\gamma(z) = \int_0^\infty \Psi_\gamma(t) e^{zt} dt, \quad z \in \mathbb{C}, 0 < \gamma < 1,$$

where

$$(2.5) \quad \Psi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\gamma n + 1 - \gamma)} = \frac{1}{2\pi i} \int_\Gamma \mu^{\gamma-1} \exp(\mu - z\mu^\gamma) d\mu$$

with Γ a contour which starts and ends at $-\infty$ and encircles the origin once counterclockwise. For more properties of the Mittag-Leffler function and function of Wright type, we refer to [12, 13].

We now turn to a short introduction to fractional powers of sectorial operators. Let A be a densely defined closed linear operator on Banach space X .

Definition 2.1. The operator A is called sectorial of angle $\omega \in [0, \pi)$ ($A \in \text{Sect}(\omega)$, in short) if

- 1) $\sigma(A)$ is contained in the closure of the sector

$$\Sigma_\omega := \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \omega\},$$

for $\omega > 0$ or $\Sigma_0 := (0, \infty)$.

- 2) For every $\omega' \in (\omega, \pi)$, $\sup\{\|zR(z, A)\| : z \in \mathbb{C} \setminus \overline{\Sigma_{\omega'}}\} < \infty$.

A family of operators $(A_\tau)_{\tau \in \Lambda}$ is called uniformly sectorial of angle $\omega \in [0, \pi)$ if $A_\tau \in \text{Sect}(\omega)$ for each τ , and $\sup\{\|zR(z, A_\tau)\| : \tau \in \Lambda, z \in \mathbb{C} \setminus \overline{\Sigma_{\omega'}}\} < \infty$.

If $0 \in \rho(A)$ for a sectorial operator A , then we can define its fractional powers as follows. For $b > 0$, define A^{-b} by

$$(2.6) \quad A^{-b} := -\frac{1}{2\pi i} \int_{\Gamma(\zeta)} \lambda^{-b} R(\lambda, A) d\lambda,$$

where the path $\Gamma(\zeta)$ runs in the resolvent set of A from $\infty e^{-i\zeta}$ to $\infty e^{i\zeta}$, while avoiding the negative real axis and the origin, and λ^b is taken as the principle branch. Noticing that $A^{-b} \in B(X)$ is injective for all $b > 0$, we can define $A^b := (A^{-b})^{-1}$ and $A^0 := I$. On the other hand, for a sectorial operator A without the assumption that $0 \in \rho(A)$, since $A + \epsilon$ is sectorial and $0 \in \rho(A + \epsilon)$, it makes sense to consider the operator $(A + \epsilon)^b$ and define the fractional powers of A by

$$A^b := s - \lim_{\epsilon \rightarrow 0^+} (A + \epsilon)^b$$

for $b > 0$ and so corresponding results for such fractional powers can be obtained by similar argument (cf. [14, 26]). We collect some basic properties of fractional powers in the following lemma.

Lemma 2.2. [14] *Let $b > 0$ and A^{-b} is defined as above. The following assertions hold.*

- (a) A^b is closed and $D(A^b) \subset D(A^c)$ for $b > c > 0$.
- (b) $A^b x = A^{b-n} A^n x$ for all $x \in D(A^n)$ and $n > b, n \in \mathbb{N}$.
- (c) Let $d > b > 0$. If $B \subset A^b$ and $D(B) = D(A^d)$, then B is closable and $\overline{B} = A^b$, where \overline{B} is the closure of B .
- (d) $\sigma(A^b) = (\sigma(A))^b$.
- (e) If $A \in \text{sect}(\omega)$ for some $\omega \in (0, \pi)$, then for every $\beta \in (0, \pi/\omega)$ the operator A^β is sectorial of angle $\beta\omega$.
- (f) If $A \in \text{sect}(\omega)$ for some $\omega \in (0, \pi)$, then the family $(A + \epsilon)_{\epsilon \geq 0}$ is uniformly sectorial of angle ω .

Finally we recall the notion of α -times resolvent families. Also here we suppose that A is a densely defined closed linear operator on X .

Definition 2.3. Let $\alpha > 0$. A family $\{S_\alpha(t)\}_{t \geq 0} \subset B(X)$ is called an α -times resolvent family generated by A if the following conditions are satisfied:

- (a) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$;
- (b) $S_\alpha(t)A \subset AS_\alpha(t)$ for $t \geq 0$;
- (c) for $x \in D(A)$, the resolvent equation

$$(2.7) \quad S_\alpha(t)x = x + \int_0^t g_\alpha(t-s)S_\alpha(s)Ax ds$$

holds for all $t \geq 0$.

Remark 2.4. Since A is densely defined and closed, it is easy to show that for all $x \in X$, $\int_0^t g_\alpha(t-s)S_\alpha(s)xds \in D(A)$ and $S_\alpha(t)x = x + A(\int_0^t g_\alpha(t-s)S_\alpha(s)xds)$.

Definition 2.5. (a) An α -times resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ is said to be bounded if there exist constants $M \geq 1$ such that $\|S_\alpha(t)\| \leq M$ for all $t \geq 0$. If A generates a bounded α -times resolvent family S_α , we will write $(A, S_\alpha) \in \mathcal{C}_\alpha(0)$ or $A \in \mathcal{C}_\alpha(0)$ for short.

(b) Let $\theta_0 \in (0, \pi/2]$ and $\omega_0 \geq 0$. An α -times resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ is called analytic of angle θ_0 for some $\theta_0 \in (0, \pi/2]$ if $S_\alpha(t)$ admits an analytic extension to the sector Σ_{θ_0} . An analytic α -times resolvent family $\{S_\alpha(z)\}_{z \in \Sigma_{\theta_0}}$ is said to be bounded if for each $\theta \in (0, \theta_0)$ there exists a constant M_θ such that

$$\|S_\alpha(z)\| \leq M_\theta, \quad z \in \Sigma_\theta.$$

If A generates a bounded analytic α -times resolvent family S_α of angle θ_0 , we will write $(A, S_\alpha) \in \mathcal{A}_\alpha(\theta_0)$ or $A \in \mathcal{A}_\alpha(\theta_0)$ for short.

Lemma 2.6. [3] *Let $0 < \alpha \leq 2$. $A \in \mathcal{C}_\alpha(0)$ if and only if $\Sigma_{\pi\alpha/2} \subset \rho(A)$ and there exists a strongly continuous function $S_\alpha : \mathbb{R}_+ \rightarrow B(X)$ such that $\|S_\alpha(t)\| \leq M$ for all $t \geq 0$ and*

$$(2.8) \quad \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad \lambda \in \Sigma_{\pi/2}$$

for all $x \in X$. Furthermore, $\{S_\alpha(t)\}_{t \geq 0}$ is the α -times resolvent family generated by A .

In the sequel we need the following important lemma on analyticity criteria for α -times resolvent families.

Lemma 2.7. *Let $\alpha \in (0, 2)$ and $\theta_0 \in (0, \min\{\frac{\pi}{2}, \frac{\pi}{\alpha} - \frac{\pi}{2}\})$. The following assertions are equivalent.*

- (a) $(A, S_\alpha) \in \mathcal{A}_\alpha(\theta_0)$.
- (b) $\Sigma_{\alpha(\frac{\pi}{2} + \theta_0)} \subset \rho(A)$, and for each $\theta \in (0, \theta_0)$, there exists a constant M_θ such that

$$\|\lambda(\lambda - A)^{-1}\| \leq M_\theta, \quad \lambda \in \Sigma_{\alpha(\frac{\pi}{2} + \theta)}.$$

- (c) $-A \in \text{Sect}(\pi - (\frac{\pi}{2} + \theta_0)\alpha)$.

The equivalence of (a) with (b) is given in [3]. (b) is equivalent to (c) by the definition of sectorial operators, which is also mentioned in Remark 3 of [16].

Remark 2.8. (a) By Lemma 2.7, $-A$ generates a bounded analytic α -times resolvent family if and only if A is sectorial of angle $\varphi < \pi - \pi\alpha/2$.

(b) If $-A$ generates a bounded α -times resolvent family, then A is sectorial of angle $\pi - \pi\alpha/2$.

Recall that if $\{S_\alpha(z)\}_{z \in \Sigma_\theta}$ is a bounded analytic α -times resolvent family with generator A , then for $t > 0$,

$$(2.9) \quad S_\alpha(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta_0}} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} d\lambda,$$

where Γ_{θ_0} is any piecewise smooth curve in $\Sigma_{\pi/2+\theta}$ going from $\infty e^{-i(\pi/2+\theta_0)}$ to $\infty e^{i(\pi/2+\theta_0)}$ for some $0 < \theta_0 < \theta$ (cf. [3, 8]).

The following subordination principle is important in the theory of fractional resolvent families, which will be extended to more general cases in Theorem 3.1.

Lemma 2.9. [3] *Let $0 < \beta < \alpha \leq 2$, $\gamma = \beta/\alpha$ and $\omega \geq 0$. If $A \in \mathcal{C}_\alpha(0)$ then $A \in \mathcal{C}_\beta(0)$ and the following representation holds*

$$S_\beta(t) = \int_0^\infty \varphi_\gamma(t, s) S_\alpha(s) ds, \quad t > 0,$$

where $\varphi_\gamma(t, s) = t^{-\gamma} \Phi_\gamma(st^{-\gamma})$ with Φ_γ defined by (2.5), in the strong sense.

3. FRACTIONAL POWERS OF GENERATORS OF FRACTIONAL RESOLVENT FAMILIES

In this section we consider the fractional generations for bounded analytic fractional resolvent families. The following theorem is our main result, which gives the answer to question (Q_5) in the Introduction.

Theorem 3.1. *Let $\alpha \in (0, 2]$ and A be sectorial of angle $\pi - \frac{\alpha}{2}\pi$ on a Banach space X , and let $0 < \gamma < 2$.*

(a) *For each $\beta \in (0, \frac{2\pi-\pi\gamma}{2\pi-\pi\alpha})$, $-A^\beta \in \mathcal{A}_\gamma(\varphi_0)$ with $\varphi_0 = \min\{\frac{\pi}{2}, -\frac{\beta}{\gamma}(\pi - \frac{\pi}{2}\alpha) + \frac{\pi}{\gamma} - \frac{\pi}{2}\}$.*

(b) *If $0 \in \rho(A)$, then the γ -times resolvent family generated by $-A^\beta$, S_γ^β , can be represented by*

$$(3.1) \quad S_\gamma^\beta(t) = \frac{1}{2\pi i} \int_{\Gamma_\omega} E_\gamma(-\mu^\beta t^\gamma) (A - \mu)^{-1} d\mu, \quad t > 0,$$

where Γ_ω is a smooth path in the resolvent of A from $\infty e^{-i\omega}$ to $\infty e^{i\omega}$, avoiding the negative axis and zero, with $\omega \in (\pi - \frac{\alpha}{2}\pi, \frac{1}{\beta}(\pi - \frac{\gamma}{2}\pi))$.

(c) *If in addition $-A$ generates a bounded α -times resolvent family S_α , then the following generalized subordination principle*

$$(3.2) \quad S_\gamma^\beta(t)x = \int_0^\infty f_{\alpha,\gamma}^\beta(t, s) S_\alpha(s)x ds, \quad t > 0,$$

holds for $x \in X$, where

$$(3.3) \quad f_{\alpha,\gamma}^\beta(t, s) = \frac{1}{2\pi i} \int_{\partial\Sigma_\omega} E_\gamma(-\mu^\beta t^\gamma) (-\mu)^{1/\alpha-1} e^{-(\mu)^{1/\alpha}s} d\mu$$

with ω as in (b), $\partial\Sigma_\omega$ is the two rays $\{\rho e^{\pm i\omega} : \rho \geq 0\}$ and $(-\rho e^{\pm i\omega})^{1/\alpha} = \rho^{1/\alpha} e^{\mp i(\pi-\omega)/\alpha}$.

Proof. (a) Since A is sectorial of angle $\pi - \frac{\pi}{2}\alpha$, by Lemma 2.2 (e), A^β is sectorial of angle $\beta(\pi - \frac{\pi}{2}\alpha)$ for $\beta \in (0, \frac{2\pi-\pi\gamma}{2\pi-\pi\alpha})$. By Lemma 2.7, $-A^\beta \in \mathcal{A}_\gamma(\varphi_0)$ if and only if $A^\beta \in \text{Sect}(\pi - (\frac{\pi}{2} + \varphi_0)\gamma)$. To guarantee that $\varphi_0 > 0$, we need $\beta < \frac{2\pi-\pi\gamma}{2\pi-\pi\alpha}$.

(b) Since $A^\beta \in \text{Sect}(\beta(\pi - \frac{\pi}{2}\alpha))$, $\rho(A^\beta) \supset \mathbb{C} - \Sigma_{\beta(\pi - \frac{\pi}{2}\alpha)}$. Thus $(\lambda + A^\beta)^{-1}$ exists and belongs to $B(X)$ for $\lambda \in \Sigma_{\pi-\beta(\pi - \frac{\pi}{2}\alpha)}$. Let $\omega > \pi - \frac{\pi}{2}\alpha$. Since $0 \in \rho(A)$, we can find $d > 0$ such that $\{z \in \mathbb{C} : |z| < d\} \subset \rho(A)$ and then choose Γ_ω as the union of Γ_ω^1 , Γ_ω^2 and Γ_ω^3 , where

$$\begin{aligned} \Gamma_\omega^1 &= \{re^{i\omega} : r > d\}, \\ \Gamma_\omega^2 &= \{de^{i\theta} : -\omega < \theta < \omega\}, \\ \Gamma_\omega^3 &= \{re^{-i\omega} : r > d\}. \end{aligned}$$

For $\lambda \in \Sigma_{\pi-\beta\omega}$, the function $f(\mu) = \frac{1}{\lambda+\mu^\beta}$ is analytic on Σ_ω , we can therefore define a bounded operator $f(A)$ as

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\omega} f(\mu)(A - \mu)^{-1} d\mu.$$

Since $\beta(\pi - \omega) < \beta(\pi - \frac{\pi}{2}\alpha)$, $(\lambda + A^\beta)^{-1} \in B(X)$ for $\lambda \in \Sigma_{\pi-\beta\omega}$. It is routine to show that for such $\lambda \in \Sigma_{\pi-\beta\omega}$,

$$(3.4) \quad (\lambda + A^\beta)^{-1} = f(A) = \frac{1}{2\pi i} \int_{\Gamma_\omega} (\lambda + \mu^\beta)^{-1} (A - \mu)^{-1} d\mu.$$

Now take β and φ_0 as in (a). Since $-A^\beta \in \mathcal{A}_\gamma(\varphi_0)$, for $0 < \delta < \varphi_0$, choose $\omega < \frac{1}{\beta}[\pi - (\frac{\pi}{2} + \delta)\gamma]$ such that when $\lambda \in \Gamma_{\frac{\pi}{2}+\delta}$ then $\lambda^\gamma \in \Sigma_{\pi-\beta\omega}$. Thus by (2.9), (3.4) and Fubini's theorem we have

$$\begin{aligned} S_\gamma^\beta(t) &= \frac{1}{2\pi i} \int_{\Gamma_{\frac{\pi}{2}+\delta}} e^{\lambda t} \lambda^{\gamma-1} (\lambda^\gamma + A^\beta)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\frac{\pi}{2}+\delta}} e^{\lambda t} \lambda^{\gamma-1} \left(\frac{1}{2\pi i} \int_{\Gamma_\omega} (\lambda^\gamma + \mu^\beta)^{-1} (A - \mu)^{-1} d\mu \right) d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_\omega} \left(\frac{1}{2\pi i} \int_{\Gamma_{\frac{\pi}{2}+\delta}} e^{\lambda t} \lambda^{\gamma-1} (\lambda^\gamma + \mu^\beta)^{-1} d\lambda \right) (A - \mu)^{-1} d\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma_\omega} E_\gamma(-\mu^\beta t^\gamma) (A - \mu)^{-1} d\mu. \end{aligned}$$

(c) We first assume that $0 \in \rho(A)$ and $A \in \mathcal{C}_\alpha(0)$. Let Γ_ω be as in (b). By (b), (2.8) and Fubini's theorem, for $x \in X$,

$$\begin{aligned} S_\gamma^\beta(t)x &= \frac{1}{2\pi i} \int_{\Gamma_\omega} E_\gamma(-\mu^\beta t^\gamma) (A - \mu)^{-1} x d\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma_\omega^1 \cup \Gamma_\omega^3} E_\gamma(-\mu^\beta t^\gamma) \left[-(-\mu)^{1/\alpha-1} \int_0^\infty e^{-(-\mu)^{1/\alpha}s} S_\alpha(s) x ds \right] d\mu \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma_\omega^2} E_\gamma(-\mu^\beta t^\gamma) (A - \mu)^{-1} x d\mu \\ &= \int_0^\infty \left[-\frac{1}{2\pi i} \int_{\Gamma_\omega^1 \cup \Gamma_\omega^3} E_\gamma(-\mu^\beta t^\gamma) (-\mu)^{1/\alpha-1} e^{-(-\mu)^{1/\alpha}s} d\mu \right] S_\alpha(s) x ds \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma_\omega^2} E_\gamma(-\mu^\beta t^\gamma) (A - \mu)^{-1} d\mu \end{aligned}$$

The integration on Γ_ω^2 converges to 0 if $d \rightarrow 0$ since $0 \in \rho(A)$. Moreover, since $|\arg(\mu^\beta t^\gamma)| < \pi - \frac{\gamma}{2}\pi$, by (2.4) the integration on $\Gamma_\omega^1 \cup \Gamma_\omega^3$ is absolutely convergent if $d \rightarrow 0$. So letting $d \rightarrow 0$, we get $S_\gamma^\beta(t)x = \int_0^\infty f_{\alpha,\gamma}^\beta(t, s) S_\alpha(s) x ds$ with

$$f_{\alpha,\gamma}^\beta(t, s) = -\frac{1}{2\pi i} \int_{\partial\Sigma_\omega} E_\gamma(-\mu^\beta t^\gamma) (-\mu)^{\frac{1-\alpha}{\alpha}} e^{-(-\mu)^{\frac{1}{\alpha}}s} d\mu.$$

Thus (3.2) holds for $-A \in \mathcal{C}_\alpha(0)$ with $0 \in \rho(A)$.

Next we show that (3.2) holds when $0 \notin \rho(A)$ and $-A$ generates a bounded analytic α -times resolvent family. For $\varepsilon > 0$, $0 \in \rho(A + \varepsilon)$ and $(A + \varepsilon)_{\varepsilon \geq 0}$ is

uniformly sectorial of angle $\pi - \frac{\alpha}{2}\pi$ by Lemma 2.2(f). By (b), the γ -times resolvent family, ${}_{\varepsilon}S_{\gamma}^{\beta}$, generated by $-(A + \varepsilon)^{\beta}$ is given by

$$(3.5) \quad {}_{\varepsilon}S_{\gamma}^{\beta}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\omega}} E_{\gamma}(-\mu^{\beta}t^{\gamma})(A + \varepsilon - \mu)^{-1}d\mu, \quad t > 0,$$

since $(A + \varepsilon - \mu)^{-1} \rightarrow (A - \mu)^{-1}$ as $\varepsilon \rightarrow 0$, by (2.4) and Lebesgue's dominated convergence theorem ${}_{\varepsilon}S_{\gamma}^{\beta}(t) \rightarrow S_{\gamma}^{\beta}(t)$ as $\varepsilon \rightarrow 0$ for every $t \geq 0$. On the other hand, by the first step we can represent ${}_{\varepsilon}S_{\gamma}^{\beta}(t)$ by

$$(3.6) \quad {}_{\varepsilon}S_{\gamma}^{\beta}(t)x = \int_0^{\infty} f_{\alpha, \gamma}^{\beta}(t, s) {}_{\varepsilon}S_{\alpha}(s)x ds, \quad t > 0,$$

where ${}_{\varepsilon}S_{\alpha}(s)$ is the α -times resolvent family generated by $-(A + \varepsilon)$. Since ${}_{\varepsilon}S_{\alpha}(s)$ is uniformly bounded and $(A + \varepsilon - \mu)^{-1} \rightarrow (A - \mu)^{-1}$ as $\varepsilon \rightarrow 0$, by the approximation theorem for α -times resolvent family (Theorem 4.2 in [23]) one has ${}_{\varepsilon}S_{\alpha}(s) \rightarrow S_{\alpha}(s)$ in strong sense for every $s \geq 0$. Note that $f_{\alpha, \gamma}^{\beta}(t, \cdot)$ is absolutely integrable by (2.4), by letting ε to 0 in (3.6) we obtain (3.2).

Finally we show that (3.2) holds when $0 \notin \rho(A)$ and $-A \in \mathcal{C}_{\alpha}(0)$. For every $\alpha' < \alpha$, $-A$ generates a bounded analytic α' -times resolvent family by (a) or Lemma 2.9, so by our second step we have for every $x \in X$,

$$S_{\gamma}^{\beta}(t)x = \int_0^{\infty} f_{\alpha', \gamma}^{\beta}(t, s) S_{\alpha'}(s)x ds, \quad t > 0,$$

where $S_{\alpha'}$ is the α' -times resolvent family generated by $-A$. Since $S_{\alpha'}(t) \rightarrow S_{\alpha}(t)$ strongly by Theorem 4.5 in [23] and $f_{\alpha', \gamma}^{\beta}(t, s) \rightarrow f_{\alpha, \gamma}^{\beta}(t, s)$, (3.2) is obtained by letting α' to α . \square

Remark 3.2. (a) Note that by Remark 2.8 (b), if $A \in \mathcal{C}_{\alpha}(0)$, then A is sectorial of angle $\pi - \alpha\pi/2$.

(b) If $\alpha = 1$, we can shift the contour in (3.3) to Γ_{ω} , that is,

$$f_{1, \gamma}^{\beta}(t, s) = \frac{1}{2\pi i} \int_{\Gamma_{\omega}} E_{\gamma}(-\mu^{\beta}t^{\gamma})e^{\mu s}d\mu.$$

(c) If $\beta = 1$, then in the proof of (b) we do not need the assumption that $0 \in \rho(A)$. Indeed, in this case we can replace the contour Γ_{ω} by $\tilde{\Gamma}_{\omega} := \Gamma_{\omega}^1 \cup \Gamma_{\omega}^3 \cup \tilde{\Gamma}_{\omega}^2$, where $\tilde{\Gamma}_{\omega}^2 = \{de^{i\theta} : \omega < \theta < 2\pi - \omega\}$, and then

$$S_{\gamma}^1(t) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_{\omega}} E_{\gamma}(-\mu t^{\gamma})(A - \mu)^{-1}d\mu, \quad t > 0,$$

and

$$f_{\alpha, \gamma}^1(t, s) = -\frac{1}{2\pi i} \int_{\tilde{\Gamma}_{\omega}} E_{\gamma}(-\mu t^{\gamma})(-\mu)^{\frac{1-\alpha}{\alpha}} e^{-(\mu)^{\frac{1}{\alpha}}s}d\mu.$$

In particular, if $(A, S_{\alpha}) \in \mathcal{A}_{\alpha}(\theta_0)$ with $\theta_0 > 0$, then for each $\theta \in (0, \theta_0)$ and $z \in \Sigma_{\theta}$, $S_{\alpha}(z)$ has the following integrated representation:

$$(3.7) \quad S_{\alpha}(z) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_{\omega}} E_{\alpha}(\mu z^{\alpha})(\mu - A)^{-1}d\mu,$$

where $\theta \in (\pi\alpha/2, (\pi/2 + \theta_0)\alpha)$. Note that (3.7) is a Dunford integral, sometimes it will be more convenient than the identity (2.9).

(d) If $\gamma = 1$, by changing the variable μ in (3.3) to $\rho e^{i\omega}$ and $\rho e^{-i\omega}$, $0 < \rho < \infty$, one gets

$$\begin{aligned} f_{\alpha,1}^\beta(t,s) &= \frac{1}{\pi} \int_0^\infty \rho^{(1-\alpha)/\alpha} \exp\left(-s\rho^{1/\alpha} \cos(\pi-\omega)/\alpha - t\rho^\beta \sin\beta\omega\right) \\ &\quad \cdot \sin\left(t\rho^\beta \sin\beta\omega - s\rho^{1/\alpha} \sin(\pi-\omega)/\alpha + (\pi-\omega)/\alpha\right) d\rho. \end{aligned}$$

As consequences of Theorem 3.1 and Remark 3.2 we have the following results, which give positive answers to questions $(Q_1) - (Q_4)$.

Corollary 3.3. *The following assertions hold.*

(a) *If $(-A, S_1) \in \mathcal{C}_1(0)$ then $-A^\alpha \in \mathcal{A}_1(\frac{\pi}{2}(1-\alpha))$ for each $\alpha \in (0, 1)$. Moreover, the C_0 -semigroup generated by $-A^\alpha$ is given by*

$$\int_0^\infty p_\alpha(t,s) S_1(s) ds, \quad t > 0$$

where $\widehat{p_\alpha(t, \cdot)}(\lambda) := \int_0^\infty e^{-\lambda s} p_\alpha(t,s) ds = e^{-\lambda^\alpha t}$ and

$$(3.8) \quad p_\alpha(t,s) = \frac{1}{\pi} \int_0^\infty \exp(sp \cos \theta - t\rho^\alpha \cos \alpha \theta) \cdot \sin(sp \sin \theta - t\rho \sin \alpha \theta + \theta) d\rho,$$

for $\pi/2 < \theta < \pi$.

(b) *If $(-A, S_\alpha) \in \mathcal{C}_\alpha(0)$ then $-A \in \mathcal{A}_\beta(\min\{\pi, (\alpha/\beta - 1)\pi/2\})$ for each $\beta \in (0, \alpha)$. Moreover, the β -times resolvent family generated by $-A$ is given by*

$$\int_0^\infty \varphi_{\beta/\alpha}(t,s) S_\alpha(s) ds, \quad t > 0$$

where $\widehat{\varphi_\gamma(\cdot, s)}(\lambda) = \lambda^{\gamma-1} e^{-\lambda^\gamma s}$, $\widehat{\varphi_\gamma(t, \cdot)}(\lambda) = E_\gamma(-\lambda t^\gamma)$ for $0 < \gamma < 1$ and

$$\begin{aligned} \varphi_\gamma(t,s) &= \frac{1}{\pi} \int_0^\infty \rho^{\gamma-1} \exp\left(-s\rho^\gamma \cos \gamma(\pi-\theta) - t\rho \cos \theta\right) \\ &\quad \cdot \sin\left(t\rho \sin \theta - s\rho^\gamma \sin \gamma(\pi-\theta) + \gamma(\pi-\theta)\right) d\rho \end{aligned}$$

for $\theta \in (\pi - \frac{\pi}{2\gamma}, \pi/2)$.

(c) *If $(-A, S_1) \in \mathcal{C}_1(0)$ then $-A^\alpha \in \mathcal{A}_\alpha(\min\{\frac{\pi}{\alpha} - \pi, \pi/2\})$ for each $\alpha \in (0, 1)$. Moreover, the α -times resolvent family generated by $-A^\alpha$ is given by*

$$\int_0^\infty f_{1,\alpha}^\alpha(t,s) S_1(s) ds, \quad t > 0$$

where $\widehat{f_{1,\alpha}^\alpha(t, \cdot)}(\lambda) = E_\alpha(-\lambda^\alpha t^\alpha)$ and $f_{1,\alpha}^\alpha(t,s) = \int_0^\infty \varphi_\alpha(t,\tau) p_\alpha(\tau,s) d\tau$.

(d) *If $(-A, S_\alpha) \in \mathcal{C}_\alpha(0)$ for some $\alpha \in (1, 2]$ then $-A^{1/\alpha} \in \mathcal{A}_1(\pi - \pi/\alpha)$. Moreover, the C_0 -semigroup generated by $-A^{1/\alpha}$ is given by*

$$\int_0^\infty f_{\alpha,1}^{1/\alpha}(t,s) S_\alpha(s) ds, \quad t > 0$$

where

$$\begin{aligned} f_{\alpha,1}^{1/\alpha}(t,s) &= \frac{\alpha}{\pi} \int_0^\infty \exp\left(-s\rho \cos(\pi-\theta)/\alpha - t\rho \cos \theta/\alpha\right) \\ &\quad \cdot \sin\left(t\rho \sin \theta/\alpha - t\rho \sin(\pi-\theta)/\alpha + (\pi-\theta)/\alpha\right) d\rho \end{aligned}$$

for $\theta \in (\pi - \frac{\alpha\pi}{2}, \alpha\pi/2)$ and $f_{\alpha,1}^{1/\alpha}(t, s) = \int_0^\infty p_{1/\alpha}(t, \tau) \varphi_{1/\alpha}(\tau, s) d\tau$.

(e) If $(-A, S_\alpha) \in \mathcal{C}_\alpha(0)$ for some $\alpha \in (0, 2]$ then $-A^{1/2} \in \mathcal{A}_{\alpha/2}(\pi/2)$. Moreover, the $\alpha/2$ -times resolvent family generated by $-A^{1/2}$ is given by

$$(3.9) \quad \frac{\alpha}{\pi} t^{\frac{\alpha}{2}} \int_0^\infty \frac{s^{\frac{\alpha}{2}-1}}{s^\alpha + t^\alpha} S_\alpha(s) ds, \quad t > 0.$$

(f) If $(-A, S_\alpha) \in \mathcal{C}_\alpha(0)$ for some $\alpha \in (0, 2)$ then $-A^\beta \in \mathcal{A}_\alpha(\min\{(\frac{\pi}{\alpha} - \frac{\pi}{2})(1 - \beta), \pi/2\})$ for $\beta \in (0, 1)$.

Proof. (a) follows from Remark 3.2 (a), (b) and (d).

(b) By Remark 3.2 (c), (2.1), Fubini's theorem and Cauchy's integral formula,

$$\begin{aligned} f_{\alpha,\beta}^1(t, s) &= -\frac{1}{2\pi i} \int_{\tilde{\Gamma}_\omega} E_\beta(-\mu t^\beta) (-\mu)^{\frac{1-\alpha}{\alpha}} e^{-(\mu)^{\frac{1}{\alpha}} s} d\mu \\ &= \int_{\tilde{\Gamma}_\omega} \left(\frac{1}{2\pi i} \int_{\Gamma_{\frac{\pi}{2}+\delta}} e^{\lambda t} \lambda^{\beta-1} (\lambda^\beta + \mu)^{-1} d\lambda \right) (-\mu)^{\frac{1-\alpha}{\alpha}} e^{-(\mu)^{\frac{1}{\alpha}} s} d\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\frac{\pi}{2}+\delta}} e^{\lambda t} \lambda^{\beta-1} \left(\int_{\tilde{\Gamma}_\omega} (\lambda^\beta + \mu)^{-1} (-\mu)^{\frac{1-\alpha}{\alpha}} e^{-(\mu)^{\frac{1}{\alpha}} s} d\mu \right) d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\frac{\pi}{2}+\delta}} e^{\lambda t} \lambda^{\beta-1} \lambda^{\beta(\frac{1-\alpha}{\alpha})} e^{-\lambda^{\beta/\alpha} s} d\lambda \\ &= \varphi_{\beta/\alpha}(t, s). \end{aligned}$$

And the last identity follows from Remark 3.2 (d) and by noting that $\varphi_\gamma(t, s) = f_{1/\gamma,1}^1(t, s)$.

(c) By Remark 3.2 (b), for $\lambda > 0$,

$$\begin{aligned} \int_0^\infty e^{-\lambda s} f_{1,\alpha}^\alpha(t, s) ds &= \int_0^\infty e^{-\lambda s} \left(\frac{1}{2\pi i} \int_{\Gamma_\omega} E_\alpha(-\mu^\alpha t^\alpha) e^{\mu s} d\mu \right) ds \\ &= \frac{1}{2\pi i} \int_{\Gamma_\omega} E_\alpha(-\mu^\alpha t^\alpha) \left(\int_0^\infty e^{-\lambda s} e^{\mu s} ds \right) d\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma_\omega} \frac{E_\alpha(-\mu^\alpha t^\alpha)}{\lambda - \mu} d\mu = E_\alpha(-\lambda^\alpha t^\alpha) \end{aligned}$$

holds by Cauchy's integral formula and (2.4). The last statement follows from the calculation of the Laplace transform of the function $\int_0^\infty \varphi_\alpha(t, \tau) p_\alpha(\tau, \cdot) d\tau$.

(d) The representation of $f_{\alpha,1}^{1/\alpha}$ follows from Remark 3.2 (d). By (b), the C_0 -semigroup generated by $-A$ is given by $T(t) = \int_0^\infty \varphi_{1/\alpha}(t, s) S_\alpha(s) ds$; and then by (a), the $(1/\alpha)$ -times resolvent family generated by $-A^{1/\alpha}$ is given by $\int_0^\infty p_{1/\alpha}(t, s) T(s) ds$.

(f) and the first part of (e) are immediate consequences of Theorem 3.1. It remains to prove the subordination formulas (3.9). Indeed, let $S_{\alpha/2}$ be the $\alpha/2$ -times resolvent family generated by $-A^{1/2}$. Since $\alpha/2 < 1$, by (5.24) in [26],

$$(3.10) \quad \left(\lambda^{\alpha/2} + A^{1/2} \right)^{-1} = \frac{1}{\pi} \int_0^\infty \frac{\mu^{1/2}}{\mu + \lambda^\alpha} (\mu + A)^{-1} d\mu.$$

Therefore, it follows from (2.8), (2.9) (3.10), and Fubini's theorem that

$$\begin{aligned}
S_{\alpha/2}(t) &= \frac{1}{2\pi i} \int_{\Gamma_{\theta_0}} e^{\lambda t} \lambda^{\alpha/2-1} \left(\lambda^{\alpha/2} + A^{1/2} \right)^{-1} d\lambda \\
&= \frac{1}{2\pi i} \int_{\Gamma_{\theta_0}} e^{\lambda t} \lambda^{\alpha/2-1} \left(\frac{1}{\pi} \int_0^\infty \frac{\mu^{1/2}}{\mu + \lambda^\alpha} ((\mu^{1/\alpha})^\alpha + A)^{-1} d\mu \right) d\lambda \\
&= \frac{1}{2\pi i} \int_{\Gamma_{\theta_0}} e^{\lambda t} \lambda^{\alpha/2-1} \left(\frac{1}{\pi} \int_0^\infty \frac{\mu^{1/\alpha-1/2}}{\mu + \lambda^\alpha} \int_0^\infty e^{-s\mu^{1/\alpha}} S_\alpha(s) ds d\mu \right) d\lambda \\
&= \frac{1}{2\pi i} \int_{\Gamma_{\theta_0}} e^{\lambda t} \left(\frac{\alpha}{\pi} \int_0^\infty S_\alpha(s) ds \int_0^\infty e^{-s\nu} \frac{\lambda^{\alpha/2-1} \nu^{\alpha/2}}{\nu^\alpha + \lambda^\alpha} d\nu \right) d\lambda \\
&= \frac{\alpha}{\pi} \int_0^\infty S_\alpha(s) ds \int_0^\infty e^{-s\nu} \left(\frac{1}{2\pi i} \int_{\Gamma_{\theta_0}} e^{\lambda t} \frac{\lambda^{\alpha/2-1} \nu^{\alpha/2}}{\nu^\alpha + \lambda^\alpha} d\lambda \right) d\nu \\
&= \frac{\alpha}{\pi} t^{\alpha/2} \int_0^\infty \frac{s^{\frac{\alpha}{2}-1}}{s^\alpha + t^\alpha} S_\alpha(s) ds, \quad t > 0.
\end{aligned}$$

□

Remark 3.4. (a) (3.8) is the formula (11) in [32]. Note that $p_{1/2}(t, s) = \frac{te^{-t^2/4s}}{2\sqrt{\pi s^{3/2}}}$ (see Lemma 1.6.7 in [2]).

(b) By Corollary 3.3 (b), we obtain the subordinate principle (Theorem 3.1 in [3]) for bounded fractional resolvent families. The formula is also applied to exponentially bounded fractional resolvent families by small modification, since we do not need the fractional power here. By Lemma 1.6.7 in [2] $\varphi_{1/2}(t, s) = \frac{e^{-s^2/4t}}{\sqrt{\pi t}}$.

(c) By Corollary 3.3 (e), if A generates a bounded C_0 -semigroup $T(t)$, then the $1/2$ -times resolvent family generated by $-(-A)^{1/2}$ is given by $\frac{\sqrt{t}}{\pi} \int_0^\infty \frac{T(s)}{(t+s)s^{1/2}} ds$.

(d) By Corollary 3.3 (e), if A generates a bounded cosine function $C(t)$, then the C_0 -semigroup generated by $-(-A)^{1/2}$ is given by $\frac{2t}{\pi} \int_0^\infty \frac{C(s)}{t^2+s^2} ds$. See also Lemma 2.1 in [9].

The following results for generators of analytic fractional resolvent families can be proved similarly as the proof of Theorem 3.1 (a) by using Lemma 2.7.

Proposition 3.5. *The following assertions hold:*

(a) If $-A \in \mathcal{A}_1(\theta_0)$ for some $\theta_0 \in (0, \frac{\pi}{2}]$, then $-A^\alpha \in \mathcal{A}_1(\frac{\pi}{2} - (\frac{\pi}{2} - \theta_0)\alpha)$ for each $\alpha \in (0, \frac{\pi}{\pi-2\theta_0})$.

(b) If $-A \in \mathcal{A}_1(\theta_0)$ for some $\theta_0 \in (0, \frac{\pi}{2}]$, then $-A^\alpha \in \mathcal{A}_\alpha(\min\{\frac{\pi}{\alpha} + \theta_0 - \pi, \pi/2\})$ for each $\alpha \in (0, \frac{\pi}{\pi-\theta_0})$.

(c) If $-A \in \mathcal{A}_\alpha(\theta_0)$ for some $\alpha \in (0, 2)$ and $\theta_0 \in (0, \min\{\frac{\pi}{2}, \frac{\pi}{\alpha} - \frac{\pi}{2}\}]$, then $-A \in \mathcal{A}_\gamma(\min\{\frac{\alpha}{\gamma}(\frac{\pi}{2} + \theta_0) - \frac{\pi}{2}, \frac{\pi}{2}\})$ for each $\gamma \in (0, \frac{(\pi+2\theta_0)\alpha}{\pi})$.

(d) If $-A \in \mathcal{A}_\alpha(\theta_0)$ for some $\alpha \in (0, 2)$ and $\theta_0 \in (0, \min\{\frac{\pi}{2}, \frac{\pi}{\alpha} - \frac{\pi}{2}\}]$, then $-A^{1/\alpha} \in \mathcal{A}_1(-\frac{\pi}{\alpha} + \theta_0 + \pi)$ if $\alpha \in (\frac{\pi}{\pi-\theta_0}, 2)$.

(e) If $-A \in \mathcal{A}_\alpha(\theta_0)$ for some $\alpha \in (0, 2)$ and $\theta_0 \in (0, \min\{\frac{\pi}{2}, \frac{\pi}{\alpha} - \frac{\pi}{2}\}]$, then $-A^\beta \in \mathcal{A}_\alpha(\min\{-\frac{\beta}{\alpha}\pi + \frac{\beta}{2\alpha}\pi + \beta\theta_0 + \frac{\pi}{\alpha} - \frac{\pi}{2}, \pi/2\})$ if $\beta \in (0, \frac{(2-\alpha)\pi}{2\pi-(\pi+\theta_0)\alpha})$.

Remark 3.6. Proposition 3.5 (a) improves Theorem 3.1 in [15] in that we do not need $0 \in \rho(A)$.

Example 3.7. Let $\alpha \in (0, 2)$ and $k > 0$. Set $X := L^p(\mathbb{R})$, $A := -kD_x^2$ with $D(A) = W^{2,p}(\mathbb{R})$. It is well known that $-A$ generates a bounded analytic semigroup of angle $\frac{\pi}{2}$. Thus, by Proposition 3.5 one has

- (a) $-A \in \mathcal{A}_\alpha(\min\{\frac{\pi}{\alpha} - \frac{\pi}{2}, \frac{\pi}{2}\})$ for all $\alpha \in (0, 2)$;
- (b) $-A^\alpha \in \mathcal{A}_\alpha(\min\{\frac{\pi}{\alpha} - \frac{\pi}{2}, \frac{\pi}{2}\})$ for all $\alpha \in (0, 2)$;
- (c) $-A^\alpha \in \mathcal{A}_1(\pi/2)$ for all $\alpha \in (0, \infty)$.

Example 3.8. Let $\alpha \in (0, 2)$ and $\theta \in [0, \pi)$. Set $X := L^2(0, 1)$, $B_\theta := -e^{i\theta}D_x^2$ with $D(B_\theta) = \{f \in W^{2,2}(0, 1) : f(0) = f(1) = 0\}$. It is proved that for $\frac{\pi}{2} < \theta \leq (1 - \frac{\alpha}{2})\pi$, $-B_\theta \in \mathcal{A}_\alpha(\theta_0)$ with $\theta_0 = \min\{\frac{\pi}{\alpha} - \frac{\theta}{\alpha} - \frac{\pi}{2}, \frac{\pi}{2}\}$, but does not generate any C_0 -semigroup (see Example 2.20 in [3]). However, by Corollary 3.3 (d), $-B_\theta^{1/\alpha} \in \mathcal{A}_1(\frac{\pi}{2} - \frac{\theta}{\alpha})$ for $\alpha \in (1, 2)$.

4. SOLUTIONS TO FRACTIONAL CAUCHY PROBLEMS

In this section we will consider the solutions of fractional Cauchy problems. First we give the definitions of solutions to the inhomogeneous initial value problem

$$(4.1) \quad \begin{aligned} D_t^\alpha u(t) &= Au(t) + f(t), \quad t \in (0, \tau) \\ u^{(k)}(0) &= x_k, \quad k = 0, 1, \dots, m-1, \end{aligned}$$

where $\tau \in (0, +\infty]$, $f \in L_{loc}^1([0, \tau]; X)$ and A is a closed densely defined operator on Banach space X .

Definition 4.1. A function $u(t) \in C([0, \tau]; X)$ is called a strong solution (or simply solution) of (4.1) if $u(t)$ satisfies:

(a) $u(t) \in C([0, \tau]; D(A)) \cap C^{m-1}([0, \tau]; X)$.

(b) $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} g_{k+1}x_k) \in C^m([0, \tau]; X)$.

(c) $u(t)$ satisfies Eq. (4.1).

$u(t) \in C([0, \tau]; X)$ is called a mild solution of (4.1) if $g_\alpha * u \in D(A)$ and

$$u(t) = \sum_{k=0}^{m-1} g_{k+1}(t)x_k + A(g_\alpha * u)(t) + (g_\alpha * f)(t), \quad t \geq 0.$$

Suppose that A generates an α -times resolvent family $S_\alpha(t)$, then the strong solution of (4.1) with $f = 0$ and $x_k \in D(A)$ is given by

$$u(t) = \sum_{k=0}^{m-1} (g_k * S_\alpha)(t)x_k,$$

see [3] for more details. So we now turn to the following problem

$$(4.2) \quad \begin{aligned} D_t^\alpha u(t) &= Au(t) + f(t), \quad t \in (0, \tau) \\ u^{(k)}(0) &= 0, \quad k = 0, 1, \dots, m-1. \end{aligned}$$

If u is a mild solution of (4.2), then $g_\alpha * u \in D(A)$ and $u = A(g_\alpha * u) + g_\alpha * f$. By Remark 2.4,

$$\begin{aligned} 1 * u &= (S_\alpha - A(g_\alpha * S_\alpha)) * u = S_\alpha * u - S_\alpha * A(g_\alpha * u) \\ &= S_\alpha * u - S_\alpha * u + S_\alpha * g_\alpha * f = g_\alpha * S_\alpha * f, \end{aligned}$$

which means that $g_\alpha * S_\alpha * f$ is differentiable and the mild solution is given by

$$(4.3) \quad u(t) = \frac{d}{dt}(g_\alpha * S_\alpha * f)(t), \quad t \geq 0.$$

Consequently we have

Proposition 4.2. *Let A be the generator of an α -times resolvent family S_α and let $f \in L^1_{loc}([0, \tau]; X)$. If (4.2) has a mild solution, then it is given by (4.3). And the mild solution of (4.1) is given by*

$$u(t) = \sum_{k=0}^{m-1} (g_k * S_\alpha)(t)x_k + \frac{d}{dt}(g_\alpha * S_\alpha * f)(t), \quad t \geq 0.$$

For the strong solutions of (4.2), we have

Proposition 4.3. *Let $\alpha \in (0, 2]$. Suppose that A is the generator of an α -times resolvent family S_α and let $f \in C([0, \tau]; X)$. Then the following statements are equivalent:*

- (a) (4.2) has a strong solution on $[0, \tau]$.
 - (b) $S_\alpha * f$ is differentiable on $[0, \tau]$.
 - (c) $\frac{d}{dt}(g_\alpha * S_\alpha * f)(t) \in D(A)$ for $t \in [0, \tau)$ and $A(\frac{d}{dt}(g_\alpha * S_\alpha * f)(t))$ is continuous on $[0, \tau]$.
- In the case $\alpha \in [1, 2]$, the condition (c) can be replaced by*
- (c)' $(g_{\alpha-1} * S_\alpha * f)(t) \in D(A)$ for $t \in [0, \tau)$ and $A(g_{\alpha-1} * S_\alpha * f)(t)$ is continuous on $[0, \tau]$.

Proof. The equivalence of (a), (b) and (c)' was given in [22] for the case $\alpha \in [1, 2]$. The case $\alpha \in (0, 1)$ can be proved similarly. \square

As a corollary we have

Corollary 4.4. *Let $\alpha \in (0, 2]$. Suppose that A is the generator of an α -times resolvent family. Then (4.2) has a strong solution on $[0, \tau]$ if one of the following conditions is satisfied:*

- (a) f is continuously differentiable on $[0, \tau]$.
- (b) $\alpha \in [1, 2]$, $f(t) \in D(A)$ for $t \in [0, \tau)$ and $Af(t) \in L^1_{loc}([0, \tau]; X)$.
- (c) $\alpha \in (0, 1)$, $f(t) \in D(A)$ for $t \in [0, \tau)$ and $g_\alpha * f$ is continuously differentiable on $[0, \tau]$.

If A generates an α -times resolvent family S_α , then for $x \in D(A^n)$ by using (2.7) several times we have

$$(4.4) \quad \begin{aligned} S_\alpha(t)x &= x + (g_\alpha * S_\alpha)(t)Ax \\ &= x + (g_\alpha * 1)(t)Ax + (g_\alpha * (g_\alpha * S_\alpha))(t)A^2x \\ &= x + g_{\alpha+1}(t)Ax + (g_{2\alpha} * S_\alpha)(t)A^2x \\ &= \dots \\ &= x + g_{\alpha+1}(t)Ax + \dots + g_{(n-1)\alpha+1}(t)A^{n-1}x + (g_{n\alpha} * S_\alpha)(t)A^n x, \end{aligned}$$

which leads to

Lemma 4.5. *If A generates an α -times resolvent family S_α , then for $x \in D(A^n)$ with $n\alpha \geq 1$, $S_\alpha(t)x$ is differentiable and*

$$\frac{d}{dt}(S_\alpha(t)x) = \sum_{k=1}^{n-1} g_{k\alpha}(t)A^k x + (g_{n\alpha-1} * S_\alpha)(t)A^n x, \quad t > 0.$$

In particular, let $\alpha = 1/m$ with $m \in \mathbb{N}$, we obtain

Proposition 4.6. *Let $m \in \mathbb{N}$. Suppose that A generates a $(1/m)$ -times resolvent family $S_{1/m}$. Then for each $x \in D(A^m)$, $S_{1/m}(\cdot)x$ solves the fractional Cauchy problem*

$$(4.5) \quad \begin{aligned} D_t^{1/m} u(t) &= Au(t), \quad t > 0, \\ u(0) &= x, \end{aligned}$$

and the initial value problem

$$(4.6) \quad \begin{aligned} v'(t) &= A^m v(t) + \sum_{k=1}^{m-1} g_{k/m}(t) A^k x, \quad t > 0, \\ v(0) &= x. \end{aligned}$$

Remarks 4.7. (a) If A generates a C_0 -semigroup, then by the subordination principle A generates an (analytic) $(1/m)$ -times resolvent family. So Proposition 4.6 gives Theorem 3.3 in [19] immediately.

(b) Note that A^m does not necessarily generate a C_0 -semigroup when A generates a $1/m$ -resolvent family, we cannot obtain the uniqueness of solution of (4.6) without any further assumption on the operator A and a counterexample was given in [6].

For the corresponding inhomogeneous problems, we have

Proposition 4.8. *Let $m \geq 2$ be fixed. Assume that A is the generator of a $(1/m)$ -times resolvent family $S_{1/m}$, then for $x \in D(A^m)$, $f(t) \in C(\mathbb{R}_+, D(A^m))$, the function $S_{1/m}(t)x + (S_{1/m} * f)(t)$ solves the two equations:*

$$(4.7) \quad \begin{aligned} D_t^{1/m} u(t) &= Au(t) + (g_{(1-1/m)} * f)(t), \quad t > 0 \\ u(0) &= x \end{aligned}$$

and

$$(4.8) \quad \begin{aligned} v'(t) &= A^m v(t) + \sum_{k=1}^{m-1} g_{k/m}(t) A^k x + \sum_{k=0}^{m-1} (g_{k/m} * A^k f)(t), \quad t > 0 \\ v(0) &= x. \end{aligned}$$

Proof. Since $g_{1/m} * (g_{(1-1/m)} * f) = g_1 * f$ is differentiable and $f(t) \in D(A)$ for all $t \geq 0$, by Proposition 4.2 and Corollary 4.4 (c), $S_{1/m}(t)x + (S_{1/m} * f)(t)$ solves (4.7). It remains to show that it is also a solution of (4.8). By Proposition 4.6, we only need to show that $S_{1/m} * f$ is differentiable, $(S_{1/m} * f)(t) \in D(A^m)$ and

$$(S_{1/m} * f)'(t) = A^m (S_{1/m} * f)(t) + \sum_{k=0}^{m-1} (g_{k/m} * A^k f)(t), \quad t > 0.$$

This follows from (4.4) since

$$\begin{aligned} & (S_{1/m} * f)(t) \\ &= \int_0^t S_{1/m}(t-s) f(s) ds \\ &= \int_0^t \left[f(s) + \sum_{k=1}^{m-1} g_{\frac{k}{m}+1}(t-s) A^k f(s) + (g_1 * S_{1/m})(t-s) A^m f(s) \right] ds. \end{aligned}$$

□

Next we will discuss the connections between some pairs of the Cauchy problems of fractional order (not necessarily a rational number) and first order.

First, we have the following direct consequences of Corollary 3.3.

Theorem 4.9. (a) Let $\alpha \in (0, 1)$ and $-A \in \mathcal{C}_1(0)$. The fractional Cauchy problem

$$(4.9) \quad \begin{aligned} D_t^\alpha v(t) &= -Av(t), \quad t > 0, \\ v(0) &= x, \end{aligned}$$

is well-posed and its unique solution is given by

$$v(t) = \int_0^\infty \varphi_\alpha(t, s)u(s)ds, \quad t > 0,$$

for each $x \in D(A)$, where φ_α is given as in Corollary 3.3 and u is the solution to the Cauchy problem

$$(4.10) \quad \begin{aligned} u'(t) &= -Au(t), \quad t > 0, \\ u(0) &= x. \end{aligned}$$

(b) Let $\alpha \in (0, 1)$ and $-A \in \mathcal{C}_1(0)$. The fractional Cauchy problem

$$(4.11) \quad \begin{aligned} v'(t) &= -A^\alpha v(t), \quad t > 0, \\ v(0) &= x, \end{aligned}$$

is well-posed and its unique solution is given by

$$v(t) = \int_0^\infty p_\alpha(t, s)u(s)ds, \quad t > 0,$$

for each $x \in D(A)$, where p_α is given as in Corollary 3.3 and u is the solution to the Cauchy problem (4.10).

(c) Let $\alpha \in (0, 1)$ and $-A \in \mathcal{C}_1(0)$. The fractional Cauchy problem

$$(4.12) \quad \begin{aligned} D_t^\alpha v(t) &= -A^\alpha v(t), \quad t > 0, \\ v(0) &= x, \end{aligned}$$

is well-posed and its unique solution is given by

$$v(t) = \int_0^\infty f_{1,\alpha}^\alpha(t, s)u(s)ds, \quad t > 0,$$

for each $x \in D(A^\alpha)$, where $f_{1,\alpha}^\alpha$ is given as in Corollary 3.3 and u is the solution to the Cauchy problem (4.10).

(d) Let $\beta \in (1, 2]$ and $-A \in \mathcal{C}_\beta(0)$. The Cauchy problem (4.10) is well-posed and its unique solution is given by

$$u(t) = \int_0^\infty \varphi_{1/\beta}(t, s)v(s)ds, \quad t > 0,$$

for each $x \in D(A)$, where v is the solution to the fractional Cauchy problem

$$(4.13) \quad \begin{aligned} D_t^\beta v(t) &= -Av(t), \quad t > 0, \\ v(0) &= x, \quad v'(0) = 0. \end{aligned}$$

(e) Let $\beta \in (1, 2]$ and $-A \in \mathcal{C}_\beta(0)$. The Cauchy problem

$$(4.14) \quad \begin{aligned} u'(t) &= -A^{1/\beta}u(t), \quad t > 0, \\ u(0) &= x, \end{aligned}$$

is well-posed and its unique solution is given by

$$u(t) = \int_0^\infty f_{\beta,1}^{1/\beta}(t,s)v(s)ds, \quad t > 0,$$

for each $x \in D(A^{1/\beta})$, where $f_{\beta,1}^{1/\beta}$ is given as in Corollary 3.3 and v is the solution to the fractional Cauchy problem (4.13).

Remark 4.10. (a) In Theorem 4.9 (a) and (c), if A generates an analytic C_0 -semigroup, then the restriction on α can be relaxed by using Proposition 3.5.

(b) By using the generalized subordination principle in Theorem 3.1 and Proposition 4.2 one can also consider the inhomogeneous fractional Cauchy problems.

Remark 4.11. The results in Theorem 4.9 can be interpreted in terms of stochastic solutions. Let $0 < \alpha < 1$ and X be a Markov process with a semigroup $T(t)f(x) = \mathbb{E}(f(X(t)))$ generated by $-A$ and let $E(t) := \inf\{x > 0 : D(t) > t\}$ be the inverse or hitting time process of the stable subordinator $D(t)$, independent of X , with $\mathbb{E}(e^{-sD(t)}) = e^{-ts^\alpha}$. If u is a solution to the problem

$$(4.15) \quad u'(t) = -Au(t); \quad u(0) = f(x),$$

then

(a) the problem

$$D_t^\alpha v(t) = -Av(t); \quad v(0) = f(x),$$

has a unique solution given by

$$v(t) = \mathbb{E}(f(X(E(t)))) = \int_0^\infty u(s)f_{E(t)}(s)ds,$$

where $f_{E(t)}(s)$ is the density of the inverse stable subordinator of index α (see also Theorem 3.3 in [7]);

(b) the problem

$$W'(t) = -A^\alpha W(t); \quad W(0) = f(x),$$

has a unique solution given by

$$W(t) = \mathbb{E}(f(X(D(t)))) = \int_0^\infty u(s)f_{D(t)}(s)ds,$$

where $f_{D(t)}(s)$ is the density of the stable subordinator of index α ;

(c) the problem

$$D_t^\alpha v(t) = -A^\alpha v(t); \quad v(0) = f(x),$$

has a unique solution given by

$$\begin{aligned} v(t) &= \mathbb{E}(f(X(D(E(t))))) = \int_0^\infty W(s)f_{E(t)}(s)ds \\ &= \int_0^\infty \left(\int_0^\infty u(r)f_{D(s)}(r)dr \right) f_{E(t)}(s)ds \end{aligned}$$

with W given in (b);

(d) if in addition that for some $\beta \in (1, 2]$ the fractional Cauchy problem

$$D^\beta V(t) = -AV(t); \quad V(0) = f(x), \quad V'(0) = 0$$

is well-posed, then the solution of (4.15), u , is subordinated to V by

$$u(t) = \int_0^\infty V(s)h_{E(t)}(s)ds,$$

where $h_{E(t)}(s)$ is the density of the inverse stable subordinator of index $1/\beta$;

(e) if the assumptions of (d) hold, then the solution to the Cauchy problem

$$v'(t) = -A^{1/\beta}v(t); \quad v(0) = f(x),$$

is connected to V by

$$v(t) = \int_0^\infty u(s)h_{D(t)}(s)ds = \int_0^\infty \left(\int_0^\infty V(r)h_{E(s)}(r)dr \right) h_{D(t)}(s)ds$$

where $h_{E(t)}$ is as in (d) and $h_{D(t)}(s)$ is the density of the stable subordinator of index $1/\beta$.

We end this paper with two examples.

Example 4.12. Let $\rho > 0$ and $m \in \mathbb{N}$. Consider the fractional relaxation equation (cf. [12])

$$(4.16) \quad \begin{aligned} D_t^{1/m}u(t) &= -\rho u(t), \quad t > 0, \\ u(0) &= x. \end{aligned}$$

The solution of (4.16) is given by $u(t) = xE_{1/m}(-\rho t^{1/m})$. By Proposition 4.6, $u(t)$ also solves

$$(4.17) \quad \begin{aligned} v'(t) &= (-\rho)^m v(t) + \sum_{k=1}^{m-1} g_{\frac{k}{m}}(t)(-\rho)^k x, \quad t > 0, \\ v(0) &= x. \end{aligned}$$

Note that the solution of (4.17) is unique. Therefore, the problem (4.16) is equivalent to the problem (4.17).

Example 4.13. By Theorem 4.9, the solution of the fractional diffusion equation of order $0 < \alpha \leq 1$

$$(4.18) \quad \begin{aligned} D_t^\alpha u(t, x) &= \Delta u(t, x), \quad t > 0, \\ u(0, x) &= f_0(x) \end{aligned}$$

is given by $u(t, x) = \int_0^\infty \varphi_\alpha(t, s)(T(s)f_0)(x)ds$, where T is the Gaussian semigroup generated by Δ . Since

$$T(s)f(x) = (k_s * f)(x) = (4\pi s)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4s} f_0(y)dy,$$

we have

$$u(t, x) = \int_{\mathbb{R}^n} \left[\int_0^\infty \varphi_\alpha(t, s)(4\pi s)^{-n/2} e^{-|x-y|^2/4s} ds \right] f_0(y)dy.$$

See also [31].

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